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2002 J. Phys. A: Math. Gen. 35 7865

(http://iopscience.iop.org/0305-4470/35/37/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 7865–7876 PII: S0305-4470(02)37192-0

Rational K-matrices and representations of twisted Yangians

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Received 21 May 2002
Published 4 September 2002
Online at stacks.iop.org/JPhysA/35/7865

Abstract

We describe the twisted Yangians $Y(\mathbf{g}, \mathbf{h})$ which arise as boundary remnants of Yangians $Y(\mathbf{g})$ in $1+1$D integrable field theories. We describe and extend our recent construction of the intertwiners of their representations (the rational boundary S- or 'K'-matrices) and perform a case-by-case analysis for all pairs $(\mathbf{g}, \mathbf{h})$, giving the $\mathbf{h}$-decomposition of $Y(\mathbf{g}, \mathbf{h})$-representations where possible.

PACS numbers: 11.55.Ds, 02.10.Yn, 02.20.−a, 11.30.Rd

1. Introduction

In recent work [1], we looked at what happens to Yangian ($Y(\mathbf{g})$-) invariant field theories on the half line. From the exact S-matrix point of view, we found that the classes of solutions of the reflection equation, and thereby the admissible boundary S-matrices, were in correspondence with the $\mathbf{h}$ for which $(\mathbf{g}, \mathbf{h})$ is a symmetric pair—that is, for which G/H is a symmetric space. Each class was then naturally parametrized by (possibly a finite cover of) G/H. From the field theory point of view, looking in particular at the principal (G-valued) chiral field, we found that the classically integrable boundary conditions were of two classes. In the one most naturally related to the above, the field was constrained to take values at the boundary in $H \subset G$ such that G/H was a symmetric space, or a translation of H. In [2], we found that the surviving remnant of the $Y(\mathbf{g})$ symmetry predicts precisely the reflection matrix structure calculated directly in our earlier paper.

In this paper, we extend the results of [2] on the construction of boundary S-matrices using this boundary symmetry, the 'twisted' Yangian $Y(\mathbf{g}, \mathbf{h})$. In particular, we investigate all the $(\mathbf{g}, \mathbf{h})$, case by case, and apply our techniques wherever possible to obtain the intertwiners of representations of $Y(\mathbf{g}, \mathbf{h})$ or 'K-matrices'—that is, the boundary S-matrices, up to an overall scalar factor. This enables us to list certain representations of $Y(\mathbf{g}, \mathbf{h})$ which contain, as $\mathbf{h}$-irreducible components, the fundamental representations of $\mathbf{h}$ as their highest components.

Our method is the analogue for the boundary/twisted case of the 'tensor product graph' (TPG) for the bulk case. This is a rather primitive technique, in the sense that it uses basic conditions on the $Y(\mathbf{g}, \mathbf{h})$-representations deduced from Wigner–Eckart theorem considerations to give the spectral decomposition of the K-matrix. It does *not* give explicit constructions of the $Y(\mathbf{g}, \mathbf{h})$ action, and breaks down in complex cases. This is precisely what happens in the bulk—where, indeed, the only explicit representations of $Y(\mathbf{g})$ constructed in the general realization below are those of Drinfeld's original paper [3], on certain $\mathbf{g}$-irreducible representations, and on $\mathbf{g} \oplus \mathbf{C}$ (i.e. *adjoint* $\oplus$ *singlet*).

2. Yangians $Y(\mathbf{g})$

Suppose the Lie algebra $\mathbf{g}$ to be generated by Q_0^a with structure constants $f^a{}_{bc}$ and (trivial) coproduct Δ,

$$\left[Q_0^a, Q_0^b\right] = \mathrm{i} f^a{}_{bc} Q_0^c \qquad \text{and} \qquad \Delta\left(Q_0^a\right) = Q_0^a \otimes 1 + 1 \otimes Q_0^a. \tag{2.1}$$

The Yangian [3] $Y(\mathbf{g})$ is the enveloping algebra generated by these and Q_1^a, where

$$\left[Q_0^a, Q_1^b\right] = \mathrm{i} f^a{}_{bc} Q_1^c \qquad \text{and} \qquad \Delta\left(Q_1^a\right) = Q_1^a \otimes 1 + 1 \otimes Q_1^a + \tfrac{1}{2} f^a{}_{bc} Q_0^b \otimes Q_0^c. \tag{2.2}$$

The requirement that Δ be a homomorphism fixes[1]

$$f^{d[ab}\left[Q_1^{c]}, Q_1^d\right] = \frac{\mathrm{i}}{12} f^{api} f^{bqj} f^{crk} f^{ijk} Q_0^{(p} Q_0^q Q_0^{r)} \tag{2.3}$$

where ( ) denotes symmetrization and [ ] anti-symmetrization on the enclosed indices, and indices have been raised and lowered freely with the invariant metric γ.

The Yangian may be considered as a deformation of the polynomial algebra $\mathbf{g}[z]$: with $Q_1^a = z Q_0^a$, the undeformed algebra would satisfy (2.3) with the right-hand side zero—that is, z^2 times the Jacobi identity. In $Y(\mathbf{g})$, (2.3) acts as a rigidity condition on the construction of higher Q_n^a from the Q_1^a. There is an ('evaluation') automorphism

$$L_\theta : Q_0^a \mapsto Q_0^a \qquad Q_1^a \mapsto Q_1^a + \theta \frac{c_A}{4\,\mathrm{i}\pi} Q_0^a$$

where c_A is the value of the quadratic Casimir of $\mathbf{g}$ in the adjoint representation. (We have chosen this normalization so that, in integrable quantum field theories with $Y(\mathbf{g})$ symmetry, θ is the particle rapidity.)

Thus any representation v of $Y(\mathbf{g})$ may be considered as carrying a parameter θ: the action of $Y(\mathbf{g})$ on v^θ is that of $L_\theta(Y(\mathbf{g}))$ on v^0. The ith fundamental representation v_i^θ of $Y(\mathbf{g})$ is, in general, reducible as a $\mathbf{g}$-representation, with one of its irreducible components (that with the greatest highest weight, where these are partially ordered using the simple roots) being the ith fundamental representation V_i of $\mathbf{g}$. In the simplest cases (which include all i for $\mathbf{g} = a_n$ and c_n), $v_i^\theta = V_i$ as a $\mathbf{g}$-representation, and $Q_1^a = \theta \frac{c_A}{4\mathrm{i}\pi} Q_0^a$ upon it.

One way to deduce the $\mathbf{g}$-irreducible components of the other v_i is to use the fusion procedure: one constructs $v_i^\theta \otimes v_j^{\theta'}$ using the $\mathbf{g}$-irreducible v_i, v_j of the last paragraph, and then notes that while $v_i^\theta \otimes v_j^{\theta'}$ is generally $Y(\mathbf{g})$-irreducible, for certain special values of $\theta - \theta'$ it may be $Y(\mathbf{g})$-reducible (though not fully reducible) to another fundamental representation v_k^0.

This can be seen using the tensor product graph (TPG) [4, 5]. One constructs a graph whose nodes are the $\mathbf{g}$-irreducible components of $v_i^\theta \otimes v_j^{\theta'}$, with edges between nodes U and V when Q_1^a has non-trivial action from U to V. The edge labels (whose calculation is detailed in [4]) are then the special values at which reducibility occurs. For example, let $v_1^\theta = V_1$, the vector representation of $so(N)$. The graph of $v_1^\theta \otimes v_1^{\theta'}$ is then

$$(2\lambda_1) \xrightarrow{\frac{2\,\mathrm{i}\pi}{N}} (\lambda_2) \xrightarrow{\mathrm{i}\pi} (0)$$

[1] For $\mathbf{g} \neq sl(2)$. For the general condition, see Drinfeld [3].

where we have labelled the representations by their highest weights, so that $(\lambda_i) \equiv V_i$. At $\theta = -\theta' = \mathrm{i}\pi/N$, this becomes reducible: the action of Q_1^a on V_2 no longer yields states in $(2\lambda_1)$, and we have constructed v_2^0, which decomposes as an $so(N)$-representation into $(\lambda_2) \oplus (0)$.

These decompositions for general $\mathbf{g}$ and i appeared incrementally in the literature [6, 7]; for a full enumeration for simply-laced $\mathbf{g}$, see [8]. Many further results can be deduced using the TPG for non-fundamental representations: if we remove from any TPG all the edges with a particular label, the remaining subgraphs each provide representations of $Y(\mathbf{g})$. Thus, in the above example, $Y(\mathbf{g})$-representations can be constructed whose $\mathbf{g}$-decomposition is $(\lambda_2) \oplus (0)$, $(2\lambda_1)$, $(2\lambda_1) \oplus (\lambda_2)$ and (0).

3. Twisted Yangians $Y(\mathbf{g}, \mathbf{h})$

Let $\mathbf{h} \subset \mathbf{g}$ be the subalgebra of $\mathbf{g}$ invariant under an involutive automorphism σ. We shall write $\mathbf{g} = \mathbf{h} \oplus \mathbf{k}$, so that $\mathbf{h}$ and $\mathbf{k}$ are the subspaces of $\mathbf{g}$ with σ-eigenvalues +1 and -1, respectively. We shall use $a, b, c, \ldots$ for general $\mathbf{g}$-indices, and use $i, j, k, \ldots$ for $\mathbf{h}$-indices and $p, q, r, \ldots$ for $\mathbf{k}$-indices.

We define the *twisted Yangian* [2] $Y(\mathbf{g}, \mathbf{h})$ to be the subalgebra of $Y(\mathbf{g})$ generated by

$$Q_0^i \tag{3.1}$$

and

$$\widetilde{Q}_1^p \equiv Q_1^p + \tfrac{1}{4}\left[C_2^{\mathbf{h}}, Q_0^p\right] \tag{3.2}$$

where $C_2^{\mathbf{h}} \equiv \gamma_{ij} Q_0^i Q_0^j$ is the quadratic Casimir operator of $\mathbf{g}$ (which we write $C_2^{\mathbf{g}}$) restricted to $\mathbf{h}$, with $\gamma_{ij} = f_{ia}{}^b f_{jb}{}^a$. (It follows immediately that, with adjoint action, $C_2^{\mathbf{g}}(\mathbf{k}) = 2C_2^{\mathbf{h}}(\mathbf{k})$.)

We can again consider $Y(\mathbf{g}, \mathbf{h})$ to be a deformation, this time of the subalgebra of ('twisted') polynomials in $\mathbf{g}[z]$ invariant under the combined action of σ and $z \mapsto -z$. Its defining feature is that $Y(\mathbf{g}, \mathbf{h})$ is a co-ideal subalgebra [2], $\Delta(Y(\mathbf{g}, \mathbf{h}) \subset Y(\mathbf{g}) \times Y(\mathbf{g}, \mathbf{h})$. (In an integrable-model setting, in which $Y(\mathbf{g}, \mathbf{h})$ is the symmetry of a model with boundary, this allows the boundary states to form representations of $Y(\mathbf{g}, \mathbf{h})$ while the bulk states form representations of $Y(\mathbf{g})$.) It specializes to the cases studied in [9], though the relationship between the two approaches remains to be fully explored.

The TPG may be generalized to deal with $Y(\mathbf{g}, \mathbf{h})$. In the same way that $Y(\mathbf{g})$-representations were generally $\mathbf{g}$-reducible, so $Y(\mathbf{g}, \mathbf{h})$-representations naturally form representations of the subalgebra $\mathbf{h} \subset Y(\mathbf{g}, \mathbf{h})$, and these are generally reducible. The key idea is the branching graph [2]: one considers v_i as a $\mathbf{g}$-representation and determines how this reduces further as an $\mathbf{h}$-representation. In the simplest cases, where $v_i = V_i$ is $\mathbf{g}$-irreducible, the $\mathbf{h}$-irreducible components of V_i are the nodes of the graph, and the edges connect those $\mathbf{h}$-irreducible representations (hereafter 'irreps') between which Q_0^p has non-trivial action, while the labels are constructed from the differences in $C_2^{\mathbf{h}}$ between these components—for details and many examples, see [2].

The more subtle cases are those where v_i is $\mathbf{g}$-reducible, and we rederive the branching graph here for this more general case. First recall that the $Y(\mathbf{g}, \mathbf{h})$-representations are intertwined by the K- or 'reflection' matrix $K_v(\theta) : v^\theta \to v^{-\theta}$, where v^θ is a $Y(\mathbf{g})$-representation. (In certain cases, in which non-self-conjugate $\mathbf{g}$-representations branch to self-conjugate $\mathbf{h}$-representations, v^θ may be conjugated by K [1].) Intertwining the Q_0^i (that is, from the physics point of view, their conservation in boundary scattering processes) requires that

$$K_v(\theta) Q_0^i = Q_0^i K_v(\theta)$$

(here, by Q_0^i we mean its representation on v) and thus that $K_v(\theta)$ act as the identity on **h**-irreducible components of v. So we have

$$K_v(\theta) = \sum_{W\subset V\subset v^\theta} \tau_W(\theta) P_W$$

where the sum is over **h**-irreps W into which the V branch, where V is a **g**-irreducible component of v^θ; P_W is the projector onto W.

To deduce relations among the τ_W we intertwine the $\widetilde{Q}_1^p$. Recall that, *within* a **g**-irreducible $V \subset v^\theta$, the action of Q_1^p is given by

$$Q_1^p = \theta \frac{c_A}{4\,\mathrm{i}\pi} Q_0^p$$

where c_A is the value of the quadratic Casimir of **g** in the adjoint representation, so that

$$\langle W \| K_v(\theta) \left(\theta \frac{c_A}{4\,\mathrm{i}\pi} Q_0^p + \frac{1}{4}\left[C_2^{\mathbf{h}}, Q_0^p\right] \right) \| W' \rangle = \langle W \| \left(-\theta \frac{c_A}{4\,\mathrm{i}\pi} Q_0^p + \frac{1}{4}\left[C_2^{\mathbf{h}}, Q_0^p\right] \right) K_v(\theta) \| W' \rangle$$

for $W, W' \subset V$. Thus, when the reduced matrix element $\langle W \| Q_0^p \| W' \rangle \neq 0$ we have

$$\frac{\tau_{W'}(\theta)}{\tau_W(\theta)} = [\Delta] \qquad \text{where} \quad [A] \equiv \frac{\frac{\mathrm{i}\pi A}{c_A} + \theta}{\frac{\mathrm{i}\pi A}{c_A} - \theta} \tag{3.3}$$

and $\Delta = C_2^{\mathbf{h}}(W) - C_2^{\mathbf{h}}(W')$. To find the W, W' for which $\langle W \| Q_0^p \| W' \rangle \neq 0$ we recall that **k** forms a representation K of **h** (irreducible in general but reducing into two conjugate representations of $\tilde{\mathbf{h}}$, with opposite $u(1)$ numbers, where $\mathbf{h} = \tilde{\mathbf{h}} \times u(1)$). A necessary condition for (3.3) to apply is then that $W \subset K \otimes W'$. Although not automatically sufficient, this is (as in the bulk case [5]) sufficiently constraining in simple cases to enable us to deduce K.

There are also links between W, W' which descend from *different* **g**-irreps. When $W \subset K \otimes W'$, there will generally be some (unknown) action of Q_1^p between them, but since $\langle W \| Q_0^p \| W' \rangle = 0$, we will have $\tau_W = \tau_{W'}$.

We then describe $K_v(\theta)$ by using a graph, in which the nodes are the equivalence classes of **h**-irreps quotiented by the relation $W \sim W' \Leftrightarrow W \subset K \otimes W'$ and (for $W \neq W'$) $W \subset V \neq V' \supset W'$. These classes, the (generally **h**-reducible) representations $\widetilde{W}_i$, are linked by an edge, directed from $\widetilde{W}_i$ to $\widetilde{W}_j$ and labelled by Δ_{ij}, whenever $W \subset K \otimes W'$ for any $W \subset \widetilde{W}_i$, $W' \subset \widetilde{W}_j$. (Note that the Δ_{ij} calculated from all such pairs W, W' are equal.)

For example, let v^θ be the **g**-reducible example we encountered earlier, $v_2^0 = (\lambda_2) \oplus (0)$ of $so(N)$, and let $\mathbf{h} = so(M) \times so(N-M)$. We denote irreps of **h** by (μ, ν), where μ is an $so(M)$ weight and ν an $so(N-M)$ weight. Then the graph is

$$(0, \lambda_2) \stackrel{N-2M-2}{\longrightarrow} (\lambda_1, \lambda_1) \oplus (0, 0) \stackrel{N-2M+2}{\longrightarrow} (\lambda_2, 0).$$

For $v_3^0 = (\lambda_3) \oplus (\lambda_1)$ we find similarly

$$(0, \lambda_3) \stackrel{N-2M-4}{\longrightarrow} (\lambda_1, \lambda_2) \oplus (0, \lambda_1) \stackrel{N-2M}{\longrightarrow} (\lambda_2, \lambda_1) \oplus (\lambda_1, 0) \stackrel{N-2M+4}{\longrightarrow} (\lambda_3, 0).$$

Thus we see that, at certain special values of θ, the graph truncates, and the action of $Y(\mathbf{g}, \mathbf{h})$ may be consistently restricted: v_2 to $(\lambda_2, 0)$ or $(0, \lambda_2)$, v_3 to $(\lambda_3, 0)$ or $(0, \lambda_3)$.

There are various limitations to this method which cause it to break down in cases more complex than those we shall treat. First, as commented upon in the introduction, when W, W' branch from the same U, the condition $W \subset K \otimes W'$ is necessary but not automatically sufficient for (3.3) to apply. Second, the method breaks down—just as does the TPG for the bulk case—when any W appears with multiplicity greater than 1. Third, when W, W' branch from different U, the action of Q_1^p is generally unknown, and so we do not know precisely for which W, W' we have $\langle W \| Q_1^p \| W' \rangle \neq 0$.

4. The rational K-matrices

First, a general result. Recall [3] that $\mathbf{g} \oplus \mathbf{C}$ always extend to a representation of $Y(\mathbf{g})$. This branches to $\mathbf{h} \oplus \mathbf{k} \oplus \mathbf{C}$ of (simple) $\mathbf{h}$. As a representation of $Y(\mathbf{g}, \mathbf{h})$ the branching graph is

$$\mathbf{h} \longrightarrow \mathbf{k} \oplus \mathbf{C}$$

(with label $C_2^{\mathbf{h}}(\mathbf{h}) - C_2^{\mathbf{h}}(\mathbf{k})$), and we see that $\mathbf{k} \oplus \mathbf{C}$ extends to a representation of $Y(\mathbf{g}, \mathbf{h})$. In the cases where $\mathbf{h}$ is not simple, the graph consists of a central node $\mathbf{k} \oplus \mathbf{C}$ and edges between this and each component of $\mathbf{h}$. We can only construct $\mathbf{k} \oplus \mathbf{C}$ as a representation of $Y(\mathbf{g}, \mathbf{h})$ in this way when all the factors of $\mathbf{h}$ are isomorphic.

We now proceed to a case-by-case analysis, detailing all the $(\mathbf{g}, \mathbf{h})$ and v_i for which our method yields results. In each case we give the representation K of $\mathbf{h}$ formed by $\mathbf{k}$. When $\mathbf{h}$ contains a $u(1)$ factor, we write $\mathbf{h} = \tilde{\mathbf{h}} \times u(1)$; we then give the decomposition of K into irreps of $\tilde{\mathbf{h}}$ (we do not specify the values of the $u(1)$ generator). For classical $\mathbf{g}$ it is simplest to deal with the b- and d-series together as $so(N)$, and so we treat the other classical groups in the same way, as $su(N)$ and $sp(2n)$. When we move on to the exceptional cases, we write $a_n = su(n+1)$, $b_n = so(2n+1)$, $c_n = sp(2n)$ and $d_n = so(2n)$ in the usual way. In all cases in which $\mathbf{h}$ contains a $u(1)$ or an $su(2)$ factor, there is a subtlety in calculating the normalization of its contribution to $C_2^{\mathbf{h}}$ relative to that of the rest of $\mathbf{h}$. For exceptional $\mathbf{g}$, this can most easily be found by considering the trace of the adjoint action on $\mathbf{g}$ of the simple factors' individual contributions to $C_2^{\mathbf{h}}$.[2] Note that, because (as pointed out earlier) $C_2^{\mathbf{h}}(\mathbf{k}) = \frac{1}{2}c_A$, any edge between K and $\mathbf{C}$ occurs at rapidity $\mathrm{i}\pi/2$, at the edge of the physical strip.

We first give the decomposition of the fundamental $Y(\mathbf{g})$-representations v_i into fundamental $\mathbf{g}$-representations V_i (though, for e_7 and e_8, only for those i of which we will be able to make use). Next we give the graphs for $K_{v_i}(\theta)$, where these can be computed. For the cases in which $\mathbf{h}$ has only one non-trivial simple factor, we then list the $Y(\mathbf{g}, \mathbf{h})$-representations which follow, in terms of $\mathbf{h}$- (or $\tilde{\mathbf{h}}$-) representations (λ) given in terms of their highest weights λ, where the fundamental weights are λ_i, following the Dynkin diagram conventions in the appendix. Where the weight of the top component is λ_i, we label these w_i. Throughout, $\lfloor x \rfloor$ denotes the integer part of x.

4.1. $\mathbf{g} = su(N)$

$$v_i = V_i \qquad i = 1, 2, \ldots, N-1.$$

Most of the $su(N)$ cases are contained in [1], but we include them for completeness.

4.1.1. $\mathbf{h} = su(M) \times su(N-M) \times u(1)$.

$$K = (\lambda_1, \lambda_{m-1}) \oplus (\lambda_{n-m-1}, \lambda_1).$$

The graphs for v_r, $r = 1, \ldots, = \lfloor N/2 \rfloor$ (the others follow by conjugation), are

$$(0, \lambda_r) \xrightarrow{N-2M-2(r-1)} (\lambda_1, \lambda_{r-1}) \longrightarrow \cdots (\lambda_p, \lambda_{r-p}) \xrightarrow{N-2M-2(r-1)+4p}$$
$$\cdots (\lambda_{r-1}, \lambda_1) \xrightarrow{N-2M+2(r-1)} (\lambda_r, 0).$$

Note that $(\lambda_M, 0) \equiv (0, 0)$ and $(0, \lambda_{N-M}) \equiv (0, 0)$, while $(\lambda_r, 0)$ vanishes for $r > M$ and $(0, \lambda_r)$ vanishes for $r > N-M$, causing the graph to truncate. Then

$$w_r = (\lambda_r, 0) \qquad r = 1, 2, \ldots, M-1$$
$$w_r' = (0, \lambda_r) \qquad r = 1, 2, \ldots, N-M-1.$$

[2] I would like to thank Tony Sudbery for help with this.

Many other (non-fundamental) representations can be constructed [10], and may involve graphs which are p-dimensional (in the sense that they contain nodes linked by $2p$ edges).

4.1.2. $\mathbf{h} = so(N)$.

$$K = (2\lambda_1).$$

Here there are no non-trivial graphs; we simply have the $\mathbf{g} \to \mathbf{h}$ branching rules for V_r:

$$\begin{array}{lll} w_r = V_r & \text{for} \quad r = 1, 2, \ldots, \lfloor (N-3)/2 \rfloor \quad \text{then} \\ N \text{ even:} & w_{(N-2)/2} = (\lambda_s + \lambda_{s'}) \quad w_{N/2} = (2\lambda_s) \oplus (2\lambda_{s'}) \\ N \text{ odd:} & w_{(N-1)/2} = (2\lambda_s). \end{array}$$

4.1.3. $\mathbf{h} = sp(N)$, $N = 2n$.

$$K = (\lambda_2).$$

The graph for v_r is

$$(\lambda_r) \xrightarrow{N-2(r-2)} (\lambda_{r-2}) \cdots \xrightarrow{N-2(r-2p)} (\lambda_{r-2p}) \xrightarrow{N+4-2(r-2p)} \cdots \begin{cases} (\lambda_2) \xrightarrow{N} (0) & r \text{ even} \\ (\lambda_3) \xrightarrow{N-2} (\lambda_1) & r \text{ odd} \end{cases}$$

so that

$$w_r = (\lambda_r) \oplus (\lambda_{r-2}) \oplus \cdots \oplus \begin{cases} (0) & r \text{ even} \\ (\lambda_1) & r \text{ odd.} \end{cases}$$

4.2. $\mathbf{g} = so(N)$

This is the only classical case for which the v_i are generally reducible as $so(N)$-representations: $v_i = V_i \oplus V_{i-2} \oplus \cdots \oplus V_{0/1}$ for $r = 1, 2, \ldots, [(N-3)/2]$, $v_s = V_s$, $v_{s'} = V_{s'}$.

4.2.1. $\mathbf{h} = so(M) \times so(N-M)$.

$$K = (\lambda_1, \lambda_1).$$

For $r = 1, 2, \ldots, [(N-3)/2]$, the graphs are

$$(0, r) \xrightarrow{N-2M-2(r-1)} (1, r-1) \longrightarrow \cdots (p, r-p) \xrightarrow{N-2M-2(r-1)+4p}$$
$$\cdots (r-1, 1) \xrightarrow{N-2M+2(r-1)} (r, 0)$$

(as in the $su(N)$ case), where (for $p < q$; $p \geqslant q$ is analogous) $(p, q) \equiv (\lambda_p, \lambda_q) \oplus (\lambda_{p-1}, \lambda_{q-1}) \oplus \cdots \oplus (0, \lambda_{q-p})$, and so

$$\begin{array}{ll} w_r = (\lambda_r, 0) & r = 1, 2, \ldots, [(M-3)/2] \\ w'_r = (0, \lambda_r) & r = 1, 2, \ldots, [(N-M-3)/2]. \end{array}$$

For v_s and $v_{s'}$ we have

$$v_{s^{(\prime)}} = w_{s^{(\prime)}} = (\lambda_s, \lambda_{s'}) \oplus (\lambda_{s'}, \lambda_s)$$

if $V_s \neq V_{s'}$ for either $so(M)$ or $so(N-M)$; $v_s = (\lambda_s, \lambda_s)$ if not.

4.2.2. $\mathbf{h} = su(n) \times u(1)$, $N = 2n$.

$$K = (\lambda_2) \oplus (\lambda_{n-2}).$$

We deal first with the spinor representations, distinguishing the cases $n = 2m + 1$ (in which $V_s^* = V_{s'}$, where $*$ denotes complex conjugation) from $n = 2m$ (in which $V^*_{s^{(\prime)}} = V_{s^{(\prime)}}$).

The graphs for V_s and $V_{s'}$ are respectively (noting $\lambda_n = 0$ and $(\lambda_{n-r}) = (\lambda_r)^*$)

$$(\lambda_{2m}) \xrightarrow{2(n+3-4m)} \cdots \xrightarrow{2(n-1-4q)} (\lambda_{2q}) \xrightarrow{2(n+3-4q)} \cdots \longrightarrow (\lambda_4) \xrightarrow{2(n-5)} (\lambda_2) \xrightarrow{2(n-1)} (0)$$

and

$$(\lambda_1) \xrightarrow{2(5-n)} (\lambda_3) \longrightarrow \cdots \xrightarrow{2(4q+1-n)} (\lambda_{2q+1}) \xrightarrow{2(4q+5-n)} \cdots \begin{cases} \xrightarrow{2(n-3)} (\lambda_{2m-1}) & n = 2m \\ \xrightarrow{2(n-1)} (\lambda_{2m+1}) & n = 2m+1 \end{cases}.$$

Thus we have, for $n = 2m+1$,

$$w_{2p} = \bigoplus_{r=0}^{p} (\lambda_{2r}) \qquad w_{2p+1} = \bigoplus_{r=0}^{p} (\lambda_{2r+1})$$

and their conjugates (for $p = 0, 1, 2, \ldots, m$); while, for $n = 2m$,

$$w_{2p} = \bigoplus_{r=0}^{p} (\lambda_{2r}) \oplus (\lambda_{n-2r}) \qquad p = 0, 1, \ldots, [(m-1)/2]$$

$$w_{2p+1} = \bigoplus_{r=0}^{p} (\lambda_{2r+1}) \oplus (\lambda_{n-2r-1}) \qquad p = 0, 1, \ldots, [(m-2)/2]$$

$$w_m = (\lambda_m) \oplus w_{m-2}$$

which are all self-conjugate.

Turning to the antisymmetric tensor representations, $v_1 = V_1$ branches trivially to $(\lambda_1) \oplus (\lambda_{n-1})$; then for $v_2 = V_2 \oplus 1$ the graph is

$$(\lambda_1 + \lambda_{n-1}) \xrightarrow{2} (\lambda_2) \oplus (\lambda_{n-2}) \oplus (0) \xrightarrow{2n-2} (0).$$

Thereafter, for $i = 3, 4, \ldots, n-2$, the graph becomes intractable for the reasons mentioned at the end of the last section.

4.3. $\mathbf{g} = sp(2n)$

$$v_i = V_i \qquad i = 1, 2, \ldots, n.$$

4.3.1. $\mathbf{h} = sp(2m) \times sp(2n-2m)$.

$$K = (\lambda_1, \lambda_1).$$

The graphs are as in section 4.1.1, and the w_r are therefore

$$w_r = (\lambda_r, 1) \qquad r = 1, 2, \ldots, m$$
$$w'_r = (1, \lambda_r) \qquad r = 1, 2, \ldots, n-m.$$

4.3.2. $\mathbf{h} = su(n) \times u(1)$.

$$K = (2\lambda_1) \oplus (2\lambda_{n-1}).$$

The $sp(2n) \to su(n)$ branching rule is

$$V_r = \bigoplus_{a=0}^{r} (\lambda_a + \lambda_{n-r+a})$$

from which the graph is

$$(\lambda_r) \xrightarrow{2-2r} (\lambda_{r-1} + \lambda_{n-1}) \longrightarrow \cdots \xrightarrow{4a-2-2r} (\lambda_{r-a} + \lambda_{n-a}) \xrightarrow{4a+2-2r} \cdots (\lambda_1 + \lambda_{n-r+1}) \xrightarrow{2r-2} (\lambda_{n-r}).$$

We thus have

$$w_r = (\lambda_r) \oplus (\lambda_{n-r}) \qquad r = 1, 2, \ldots, [n/2].$$

4.4. $\mathbf{g} = e_6$

$$v_1 = V_1, v_6 = V_6; v_2 = V_2 \oplus \mathbf{C}.$$

4.4.1. $\mathbf{h} = c_4$.

$$K = (\lambda_4)$$
$$v_1 \text{ or } v_6: \quad (\lambda_2)$$
$$v_2: \quad (\lambda_4) \oplus (0) \xrightarrow{2} (2\lambda_1)$$
$$v_3 \text{ or } v_5: \quad (\lambda_1 + \lambda_3) \oplus (\lambda_2) \xrightarrow{6} (2\lambda_1)$$

so that $w_2 = (\lambda_2)$, $w_4 = (\lambda_4) \oplus (0)$.

4.4.2. $\mathbf{h} = d_5 \times u(1)$.

$$K = (\lambda_4) \oplus (\lambda_5)$$
$$v_1: \quad (\lambda_1) \xleftarrow{2} (\lambda_5) \xrightarrow{10} (0)$$
$$v_2: \quad (\lambda_2) \xrightarrow{4} (\lambda_4) \oplus (\lambda_5) \oplus (0) \xrightarrow{12} (0)$$

and v_6 as v_1^*; the others are intractable.

Hence $w_1 = (\lambda_1)$, $w_2 = (\lambda_2)$, $w_4 = (\lambda_4) \oplus (0)$, $w_5 = (\lambda_5) \oplus (0)$, others unknown.

4.4.3. $\mathbf{h} = a_5 \times a_1$.

$$K = (\lambda_3, \lambda_1).$$

Here we abbreviate the (spin-$s/2$) $su(2)$ irrep $(s\lambda_1)$ to (s).

$$v_1: \quad (\lambda_4, 0) \xrightarrow{2} (\lambda_1, 1)$$
$$v_2: \quad (\lambda_5 + \lambda_1, 0) \xrightarrow{0} (\lambda_3, 1) \oplus (0, 0) \xrightarrow{8} (0, 2)$$

v_6 as v_1^*, and we can go no further.

4.4.4. $\mathbf{h} = f_4$.

$$K = (\lambda_4)$$
$$v_1: \quad (\lambda_4) \xrightarrow{12} (0)$$
$$v_2: \quad (\lambda_1) \xrightarrow{6} (\lambda_4) \oplus (0)$$

and $w_1 = (\lambda_1)$ and $w_4 = (\lambda_4)$ in addition to $\mathbf{k} \oplus \mathbf{C} = (\lambda_4) \oplus (0)$.

4.5. $\mathbf{g} = e_7$

$v_1 = V_1 \oplus \mathbf{C}$, $v_7 = V_7$, $v_2 = V_2 \oplus V_7$ are the only cases we can treat.

4.5.1. $\mathbf{h} = e_6 \times u(1)$.

$$K = (\lambda_1) \oplus (\lambda_6)$$
$$v_1: \quad (\lambda_2) \xrightarrow{6} (\lambda_1) \oplus (\lambda_6) \oplus (0) \xrightarrow{18} (0)$$

and $w_2 = (\lambda_2)$.

4.5.2. $\mathbf{h} = d_6 \times a_1$.

$$K = (\lambda_5, 1)$$
$$v_7\colon\quad (\lambda_6, 0) \xrightarrow{4} (\lambda_1, 1)$$
$$v_1\colon\quad (\lambda_2, 0) \xrightarrow{2} (\lambda_5, 1) \oplus (0, 0) \xrightarrow{14} (0, 2).$$

4.5.3. $\mathbf{h} = a_7$.

$$K = (\lambda_4)$$
$$v_7\colon\quad (\lambda_2) \oplus (\lambda_6)$$
$$v_1\colon\quad (\lambda_4) \oplus (0) \xrightarrow{2} (\lambda_1 + \lambda_7)$$
$$v_2\colon\quad (2\lambda_1) \xleftarrow{8} (\lambda_1 + \lambda_5) \oplus (\lambda_2) \oplus (\lambda_3 + \lambda_7) \oplus (\lambda_6) \xrightarrow{8} (2\lambda_7)$$

and $w_4 = (\lambda_4) \oplus (0) (= \mathbf{k} \oplus \mathbf{C})$.

4.6. $\mathbf{g} = e_8$

$$v_8 = V_8 \oplus \mathbf{C}.$$

4.6.1. $\mathbf{h} = e_7 \times a_1$.

$$K = (\lambda_7, 1)$$
$$v_8\colon\quad (\lambda_1, 0) \xrightarrow{6} (\lambda_7, 1) \oplus (0, 0) \xrightarrow{26} (0, 0)$$

and $w_1 = (\lambda_1, 0)$.

4.6.2. $\mathbf{h} = d_8$.

$$K = (\lambda_7)$$
$$v_8\colon\quad (\lambda_7) \oplus (0) \xrightarrow{2} (\lambda_2)$$
$$v_1\colon\quad (\lambda_4) \oplus (\lambda_7) \oplus (0) \xrightarrow{2} (\lambda_1 + \lambda_8) \oplus (\lambda_2) \xrightarrow{14} (2\lambda_1).$$

4.7. $\mathbf{g} = f_4$

$$v_4 = V_4,\ v_1 = V_1 \oplus \mathbf{C}.$$

4.7.1. $\mathbf{h} = b_4$.

$$K = (\lambda_4)$$
$$v_4\colon\quad (\lambda_1) \xleftarrow{1} (\lambda_4) \xrightarrow{9} (0)$$
$$v_1\colon\quad (\lambda_2) \xrightarrow{5} (\lambda_4) \oplus (0)$$

so $w_1 = (\lambda_1)$, $w_2 = (\lambda_2)$, $w_4 = (\lambda_4) \oplus (0)$.

4.7.2. $\mathbf{h} = c_3 \times a_1$.

$$K = (\lambda_3, 1)$$
$$v_4\colon\quad (\lambda_2, 0) \xrightarrow{1} (\lambda_1, 1)$$
$$v_1\colon\quad (2\lambda_1, 0) \xleftarrow{1} (\lambda_3, 1) \oplus (0, 0) \xrightarrow{5} (0, 2).$$

4.8. $\mathbf{g} = g_2$

$$v_1 = V_1,\ v_2 = V_2 \oplus \mathbf{C}.$$

4.8.1. $\mathbf{h} = a_1 \times a_1$.

$$K = (3, 1)$$
$$v_1: \quad (2, 0) \xrightarrow{-2/3} (1, 1)$$
$$v_2: \quad (2, 0) \xleftarrow{8/3} (3, 1) \oplus (0, 0) \xrightarrow{0} (0, 2).$$

5. *K*-matrices and the magic square

The Freudenthal–Tits magic square (see [11] and references therein) provides a remarkable construction, based on division algebras, of the exceptional Lie algebras. We recall here only that it may be written as

$m =$	1	2	4	8
	a_1	a_2	c_3	f_4
	a_2	$a_2 \times a_2$	a_5	e_6
	c_3	a_5	d_6	e_7
	f_4	e_6	e_7	e_8

It is no coincidence that, if we write $g_m^{(i)}$ for the appropriate entry of the ith row, then $g_m^{(4)} \big/ \big(g_m^{(3)} \times a_1\big)$, $g_m^{(3)} \big/ \big(g_m^{(2)} \times u(1)\big)$ and $g_m^{(2)} \big/ g_m^{(1)}$ are all symmetric spaces—this is fundamental in the construction of the square. The parameter m is the order of an underlying division algebra (real or complex numbers, quaternions or octonions, here as a derivation algebra; the rows are labelled in the same way by triality algebras). For each of $i = 1, 2, 3$ the dimension of the corresponding representation K is a linear function of m, respectively $3m + 2$, $2(3m + 3)$ and $2(6m + 8)$.

It has already been noted [12] that the R-matrices (the solutions of the bulk Yang–Baxter equation) in these distinguished representations have a uniform graph structure for each i, with the graph labels having a simple linear dependence on m. We note here that the same is true of the K-matrices we have constructed: for $g_m^{(2)} \big/ g_m^{(1)}$, there are K-matrices

$$\mathbf{h} \xrightarrow{2m-4} \mathbf{k} \oplus \mathbf{C} \qquad \mathbf{k} \xrightarrow{3m} \mathbf{C}$$

while for $g_m^{(3)} \big/ \big(g_m^{(2)} \times u(1)\big)$ we have

$$\mathbf{h} \xrightarrow{m-2} \mathbf{k} \oplus \mathbf{C} \xrightarrow{2m+2} \mathbf{C}.$$

Finally, for $g_m^{(4)} \big/ \big(g_m^{(3)} \times a_1\big)$ there is

$$(\mathbf{h}, 0) \xrightarrow{m-2} (\mathbf{k}, 1) \oplus \mathbf{C} \xrightarrow{3m+2} (0, 2)$$

and, for all but $m = 8$,

$$(U, 0) \xrightarrow{m} (V, 1)$$

where V is the vector representation of $g_m^{(3)}$ and U a representation for which we have no general characterization. This even extends, just as in the bulk case, to the 'zeroth' column, in which $g^{(4)} = g_2$ for $m = -2/3$.

At this level, of course, the above is merely a nice observation, but it does suggest that it might be interesting to study Yangians and twisted Yangians—indeed, Yang–Baxter and reflection equation algebras more generally—from a division algebra point of view.

Acknowledgments

I would like to thank Tony Sudbery and Bruce Westbury for discussions, and Ben Short, Max Nazarov and Gustav Delius for helpful comments.

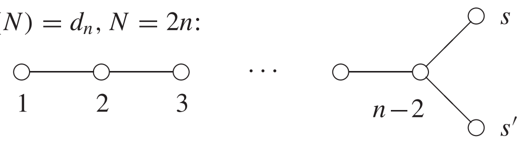
Appendix. Dynkin diagrams and conventions

$su(N) = a_n$, $N = n + 1$:

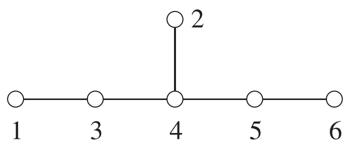
$so(N) = b_n$, $N = 2n + 1$:

$sp(2n) = c_n$:

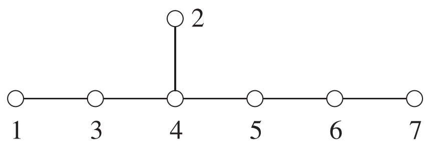
$so(N) = d_n$, $N = 2n$:



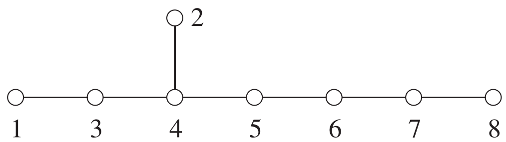
e_6:



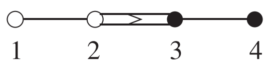
e_7:



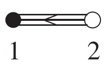
e_8:



f_4:



g_2:



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